Statistical Theory
Solutions to Exercises

Dr Nick Fieller

Department of Probability & Statistics

University of Sheffield

PAS6004/PAS373

2001/02 Semester 1

1) Exponential families

a) $f(x;\theta) = (1-\theta)\theta^x = \exp\{x\log\theta + \log(1-\theta)+0\}$

b) $f(x;\alpha,\beta) = \{\Gamma(\alpha+\beta)/\Gamma(\alpha)\Gamma(\beta)\}x^{\alpha-1}(1-x)^{\beta-1}$

$$= \exp\{\sum_{i=1}^{2} a_i(\alpha,\beta)b_i(x) + c(\alpha,\beta) + d(x)\}$$

with $a_1=\alpha-1$, $b_1=\log(x)$

$a_2=\beta-1$, $b_2=\log(1-x)$

$c=\log\Gamma(\alpha+\beta)-\log\Gamma(\alpha)-\log\Gamma(\beta)$, $d=0$

c) $f((x;\alpha,\lambda) = \{\lambda^\alpha/\Gamma(\alpha)\}e^{-\lambda x}x^{\alpha-1}$

$$= \exp\{\sum_{i=1}^{2} a_i(\alpha,\lambda)b_i(x) + c(\alpha,\lambda) + d(x)\}$$

with $a_1=-\lambda$; $b_1=x$; $a_2=\alpha-1$; $b_2=\log(x)$;

$c=\log(\lambda^\alpha/\Gamma(\alpha)$; $d=0$

d) clearly not in exponential family since values of x for which $f(x;\theta)$ is zero depends on θ. More formally, if it were in the exponential family then [in standard notation] we would need either b(x) or d(x) (functions of x ***only*** to be such that $\exp\{a(\theta)b(x)\}=0$ or $\exp\{d(x)\}=0$ when $x<\theta$, a contradiction.

2) Parts a), b) and c) all follow directly in each case from exponential family form given in Q1 above, followed by factorisation theorem (with $m_2(x_1,\ldots,x_n)=1$ in each case. For part d) either shew $lik(\theta;\underline{x})=\exp(\theta)hv(\theta,x_{min})\exp(-\Sigma x_i)$ and then use the factorization theorem, or else prove directly by shewing density of x_{min} is $f(t)=n\exp\{-n(t-\theta)\}$ and then shewing conditional density of $\underline{x}$ given x_{min} is independent of θ.

3) Noting that $\Sigma(x_i-\mu)^2=\Sigma(x_i-\bar{x})^2+n(\bar{x}-\mu)^2$ shews that $lik(\mu;\underline{x})=\exp\{-\frac{1}{2}n(\bar{x}-\mu)^2\}m_2(\underline{x})$ so $\bar{x}$ is sufficient by the factorization theorem.

4) $lik(\theta;x)=(2\pi)^{-\frac{1}{2}}\theta^{-1}\exp(-x^2/2\theta^2\} \times 1$ and so x is sufficient for θ, though not minimally sufficient since $-x$ is also sufficient; in fact $|x|$ is minimally sufficient.

5)

a) $lik(\sigma^2;x)=(2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp(-\Sigma(x_i-\mu)^2/2\sigma^2\} \times 1$ and so $\frac{1}{n}\Sigma_1^n(X_i-\mu)^2$ is sufficient for σ^2. Noting that $\Sigma(x_i-\mu)^2=\Sigma(x_i-\bar{x})^2+n(\bar{x}-\mu)^2$ and clearly $(\bar{x}-\mu)^2$ cannot be be expressed as a function of $\Sigma(x_i-\bar{x})^2$ shews that $\frac{1}{n-1}\sum_1^n(X_i-\bar{X})^2$ is not sufficient for θ^2.

b) If $Y_i=(X_i-\mu)/\sigma$ then $Y_i\sim N(0,1)$ and since $\frac{X_i-\bar{X}}{\bar{X}-\mu}=\frac{Y_i=\bar{Y}}{\bar{Y}}$ its distribution must be independent ofσ and so it is an ancilliary statistic.

6) $f(\underline{x};\theta)=\Pi f(x_i|\theta)$. If $\underline{s}=(x_{(1)},\ldots,x_{(n)})$ then $f(\underline{s};\theta)=n!\Pi f(x_{(i)}|\theta)$ provided $x_{(1)}<x_{(2)}<\ldots<x_{(n)}$ and 0 otherwise, so $f(\underline{x}|\underline{s};\theta)=(n!)^{-1}$ if $x_{(1)}<x_{(2)}<\ldots<x_{(n)}$ and 0 otherwise, and since this is independent of θ we have that $\underline{s}$ is sufficient for θ.

7) X~B(n,θ) so $E\left[\frac{X(X-1)}{n(n-1)}\right] = \frac{1}{n(n-1)}\left\{E\left[X^2\right] - E[X]\right\}$

$$= \frac{1}{n(n-1)}\left\{\text{var}(X) + \left[E[X]\right]^2 - E[X]\right\}$$

$= \{n\theta(1-\theta) + (n\theta)^2 - n\theta\}/n(n-1) = \theta^2$ and so is unbiased for θ^2. $E[X(X-1)]=n(n-1)\theta^2$ so $E[X^2/n^2]=(n-1)\theta^2/n+\theta/n\neq\theta^2$ and so X^2/n^2 is biased for θ^2

8) X_1 and X_2 are independent and each is P(θ). We have $E[X_i]=\theta=\text{var}(X_i)$ so

i) if $T_1= X_1$ then $E[T_1]=\theta$, $\text{var}(T_1)=\theta$

ii) if $T_2=½(X_1+X_2)$ then $E[T_2]=\theta$, $\text{var}(T_2)=½\theta$

iii) if $T_3=X_1X_2$ then $E[T_3]=E[X_1]E[X_2]=\theta^2$,
$\text{var}(T_3)=E[X_1^2X_2^2]-\{E[X_1]E[X_2]\}^2=\theta^2(1+\theta)^2--\theta^4=\theta^2(1+2\theta)^2$.
So efficiency of T_1 relative to T_2 is $\text{var}(T_1)/\text{var}(T_2)=50\%$. T_3 is biased. Since T_1 and T_2 are unbiased their MSEs are the same as their variances.
$\text{MSE}(T_3)=E[(X_1X_2-\theta)^2]=\theta^2(1+\theta)^2-2\theta.\theta^2+\theta^2=\theta^2(2+\theta^2)$ and so T_2 has the smallest MSE except for small values of θ near 0.

9) $X_i \sim P(\alpha_i\theta)$ so $E[X_i]=\alpha_i\theta=\text{var}(X_i)$. So

i) if $T_1=\dfrac{X_1-X_2}{\alpha_1-\alpha_2}$ then $E[T_1]=\theta$ and $\text{var}(T_1)=(\alpha_1+\alpha_2)\theta/(\alpha_1-\alpha_2)^2$

ii) if $T_2=\dfrac{X_1+X_2}{\alpha_1+\alpha_2}$ then $E[T_2]=\theta$ and $\text{var}(T_2)=\theta/(\alpha_1+\alpha_2)$

iii) if $T_3=\dfrac{X_1}{2\alpha_1}+\dfrac{X_2}{2\alpha_2}$ then $E[T_3]=\theta$ and $\text{var}(T_3)=\dfrac{1}{4}\left(\dfrac{1}{\alpha_1}+\dfrac{1}{\alpha_2}\right)\theta$

Thus T_1, T_2 and T_3 are all unbiased but $\text{var}(T_1) < \text{var}(T_2)$ and $\text{var}(T_3) < \text{var}(T_2)$ for all θ. Since they are unbiased their MSEs are the same as their variances. Note that T_1 could take negative values (i.e. outside the parameter space for θ).

10) Suppose $t(x)$ takes values t_0, t_1 and t_2 as x takes values 0, 1 and 2. Then we require that $E[T(X)]= t_0\theta^2+ t_1\theta(1-\theta)+ t_2(1-\theta) = \theta$ for all θ. Comparing coefficients of θ^r for r=0, 1 and 2 shews that we have $t_0-t_1=0$, $t_1=1$ and $t_2=0$ hence the unique unbiased estimator takes values 1, 1 and 0 as x takes values 0, 1 and 2.

11) Suppose $t(x)$ takes values t_i for x=i, i=0, 1, 2, 3. Then we require that $E[T(X)]= 2t_0\theta/3+ t_1(1-\theta)/3+ t_2(2-\theta)/3=\theta$ for all θ. So we must have $2t_0/3- t_1/3- t_2/3=1$ and $t_1/3+ 2t_2/3=0$. So either t_1 or $t_2 < 0$ or else $t_1=t_2=0$ in which case $t_0=3/2$, i.e. there is a unique class of unbiased estimators but they all take values outside the parameter space for θ.

12) With similar arguments and notation to above it is easy to shew that $t_3=0$, $t_0=1$ and $t_1=2–3t_2$ so there is not a unique unbiased estimator but there is a unique class of estimators given by these expressions. For them to take values in the parameter space we would have to require that $1/3<t_2<2/3$.

13) $E[X_i]=\theta$, $E[X_i^2]=3\theta/2$, $\text{var}(X_i)=$ (by direct calculation).

i) if $T_1=\dfrac{X_1+X_2+\ldots+X_n}{n}$ then $E[T_1]=\theta$ and $\text{var}(T_1)=\theta(3/2-\theta)/n$

ii) if $T_2=\dfrac{4n_{12}}{3n}$ then since $n_{12}\sim B(n, 3\theta/4)$ we have $E[T_2]=\theta$ and $\text{var}(T_2)=4\theta(1-\theta)/3n < \text{var}(T_1)$.

14)

a) $f(x;\theta) = \theta^2$ if $x=1$and $1-\theta^2$ if $x=0$, i.e. $f(x;\theta) = (\theta^2)^x(1-\theta^2)^{1-x}$ so

$\ell(\theta)=n\bar{x}\log\theta^2-n(1-\bar{x})\log(1-\theta^2)$ and so

$\dfrac{\partial\ell}{\partial(\theta^2)}=\dfrac{n\bar{x}}{\theta^2}-\dfrac{n(1-\bar{x}}{1-\theta^2}=\dfrac{n}{\theta^2(1-\theta^2)}(\bar{x}-\theta^2)$ and so the MLE of θ^2 is $\bar{x}$ which is unbiased for θ^2 and so of θ itself it is $\sqrt{\bar{x}}$ which must be biased.

b) $f(x;\lambda) = \{\lambda^\alpha/\Gamma(\alpha)\}e^{-\lambda x}x^{\alpha-1}$ so

$\ell(\lambda)=n\alpha\log\lambda-n\log\Gamma(\alpha)-n\lambda\bar{x}-(\alpha-1)\Sigma x_i$

$\dfrac{\partial\ell}{\partial\lambda}=\dfrac{n\alpha}{\lambda}-n\bar{x}$ and so the MLE of λ is $\alpha/\bar{x}$. This is biased, since $\bar{x}/\alpha$ is unbiased for λ^{-1} .

c) Easy to shew that $\Sigma(x_i-\mu)^2/n$ is the MLE of σ^2 and is unbiased for it. So MLE of σ is $\{\Sigma(x_i-\mu)^2/n\}^{½}$ which must be biased.