Statistical Theory
Exercises

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1. Investigate whether or not the following densities are within the exponential family:

   (a) $f(x;\theta) = (1-\theta)\theta^x$, x=0,1,2,...., (with $0<\theta<1$).

   (b) $f(x;\alpha,\beta) = \{\Gamma(\alpha+\beta)/\Gamma(\alpha)\Gamma(\beta)\}x^{\alpha-1}(1-x)^{\beta-1}$, for $0<x<1$, (with $\alpha>0$, $\beta>0$);

   (c) $f((x;\alpha,\lambda) = \{\lambda^\alpha/\Gamma(\alpha)\}e^{-\lambda x}x^{\alpha-1}$, for $x>0$, (with $\alpha>0$, $\lambda>0$);

   (d) $f(x;\theta) = \exp\{-(x-\theta)\}$, for $x\geq\theta$, (0 otherwise);

2. Assuming $X_1,X_2,...,X_n$ to be a random sample from the populations (a) — (d) in Q1 above respectively, shew that:

   (a) $\sum_1^n X_i$ is sufficient for θ.

   (b) $\left(\sum_1^n \log(X_i),\ \sum_1^n \log(1-X_i)\right)$ is sufficient for (α,β);

   (c) $\left(\sum_1^n \log(X_i),\ \sum_1^n X_i\right)$ is sufficient for (α,λ);

   (d) $X_{(1)}= \min_{1\leq i\leq n}\{X_i\}$ is sufficient for θ;

3. If $X_1,X_2,...,X_n$ is a random sample from $N(\mu,1)$, find a sufficient statistic for μ.

4. X is a single observation from $N(0,\theta)$. Is X a sufficient statistic for θ?

5. Prove that if $X_1,X_2,...,X_n$ is a random sample from $N(\mu,\sigma^2)$ *with μ known,* then $\frac{1}{n}\sum_1^n(X_i-\mu)^2$ is sufficient for σ^2 but that $\frac{1}{n-1}\sum_1^n(X_i-\bar{X})^2$ is not sufficient for σ^2. Also shew that $\dfrac{X_i-\bar{X}}{\bar{X}-\mu}$ is an ancillary statistic for each i=1,2,...,n.

6. If $X_1,X_2,\ldots,X_n$ are i.i.d with a continuous p.d.f. with parameter θ, shew that the order statistic $(X_{(1)},X_{(2)},\ldots,X_{(n)})$, where $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$, is jointly sufficient for θ.

7. In n independent replications of an experiment with probability θ of success, a success occurs in X cases. Shew that $\dfrac{X(X-1)}{n(n-1)}$ is an unbiased estimator of θ^2 and hence that $\dfrac{X^2}{n^2}$ is biased.

8. X_1 and X_2 are independent and each is $P(\theta)$. Which of the following are unbiased estimators of θ and which of the unbiased estimators is the most efficient?

   (i) X_1 (ii) $\frac{1}{2}(X_1+X_2)$ (iii) X_1X_2

   Which has the smallest mean squared error?

9. The number of flaws in a piece of material of length α from a certain loom is assumed to be a $P(\alpha\theta)$ variable, where θ is unknown. In two independently produced pieces of material from this loom of known lengths α_1 and α_2, where $\alpha_1>\alpha_2$, the numbers of flaws are X_1 and X_2. Which of the following are unbiased estimators of θ and which of the unbiased estimators is the most efficient?

   (i) $\dfrac{X_1-X_2}{\alpha_1-\alpha_2}$ (ii) $\dfrac{X_1+X_2}{\alpha_1+\alpha_2}$ (iii) $\dfrac{X_1}{2\alpha_1}+\dfrac{X_2}{2\alpha_2}$

   Which has the smallest mean squared error?

10. The discrete random variable X takes values 0, 1 and 2 with probabilities θ^2, $\theta(1-\theta)$ and $1-\theta$ respectively, where $0\leq\theta\leq1$. Shew that there is a unique unbiased estimator of θ based on one observation of X.

11. The discrete random variable X takes values 0, 1 and 2 with probabilities $2\theta/3$, $(1-\theta)/3$ and $(2-\theta)/3$ respectively, where $0\le\theta\le1$. Shew that there is a unique unbiased estimator of θ based on one observation of X. Do you have any comment on this estimator?

12. The discrete random variable X takes values 0, 1, 2 and 3 with probabilities θ^2, $\theta(1-\theta)/2$ and $3\theta(1-\theta)/2$ and $(1-\theta)^2$ respectively, where $0\le\theta\le1$. Shew that there is not a unique unbiased estimator of θ based on one observation of X. Determine the class of unbiased estimators of θ and hence the MVUE of θ.

13. $X_1,X_2,\ldots,X_n$ are independent observations of the discrete random variable X which takes values 0, 1 and 2 with probabilities $1-\frac{3}{4}\theta$, $\frac{1}{2}\theta$ and $\frac{1}{4}\theta$ respectively, where $0\le\theta\le1$. The number of replicates in which either 1 or 2 is recorded is n_{12}. Which of the following two estimators is to be preferred?

    (i) $\dfrac{X_1+X_2+\ldots+X_n}{n}$ (ii) $\dfrac{4n_{12}}{3n}$

14. Assuming that a random sample of size n is available from each of the following distributions, obtain the MLEs of the parameters:

    (a) $f(x;\theta) = \theta^2$ if x=1, $f(x;\theta) = 1-\theta^2$ if x=0;

    (b) $f(x;\lambda) = \{\lambda^\alpha/\Gamma(\alpha)\}e^{-\lambda x}x^{\alpha-1}$,for x>0,

    (with $\lambda>0$, & α ***known***);

    (c) $f(x;\sigma) = (2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\}$, with μ ***known***.

    In each case determine whether the estimator is an unbiased estimator of the parameter.

15. Obtain the MLE of θ based on a random sample of size n from a uniform distribution on $[\alpha,\alpha+\theta]$ with α known. Is this an unbiased estimator of θ?

16. In the following cases determine whether the MLEs of θ based on n independent observations are MVBU estimators

    (a) $f(x;\sigma) = (2\pi)^{-½}\sigma^{-1}\exp\{-½(x-\mu)^2/\sigma^2\}$, with μ *known*; $\theta=\sigma^2$.

    (b) $f(x;\lambda) = \{\lambda^{\alpha}/\Gamma(\alpha)\}e^{-\lambda x}x^{\alpha-1}$,for $x>0$,

    (with $\lambda>0$, & α ***known***); $\theta=\lambda^{-1}$.

    (c) $f(x;\theta) = \theta^2$ if $x=1$, $f(x;\theta) = 1-\theta^2$ if $x=0$;

17. Are the estimators obtained in Q10 & Q12 MVBU estimators?

18. $x_1,x_2,\ldots,x_n$ are independent observations of $X\sim U(0,\theta)$ & $\eta(\theta)=\theta^2$.

    (a) Find $E\left[\left(\partial\ell(\theta)/\partial\theta\right)^2\right]$ where $\ell(\theta)$ is the log likelihood of θ for the data.

    (b) Find the MLE of θ and hence of $\eta(\theta)$.

    (c) Find $E[\hat\eta(\theta)]$ and hence derive an unbiased estimator of $\eta(\theta)$ of the form $k\hat\eta(\theta)$ (=$t(x_1, x_2,\ldots,x_n)$=t say) for a suitable constant k.

    (d) Find Var(t).

    (e) Find $\{\eta'(\theta)\}^2 \Big/ E\left[\left(\partial\ell(\theta)/\partial\theta\right)^2\right]$.

    (f) Explain why the values found in (d) and (e) are in apparent contradiction to the Cramer-Rao result.

19. Which of the MLEs of θ in Q14, Q16 and Q18 are consistent?

20. Are MVBU estimators based on n independent observations always consistent?

21. Give an example of an unbiased estimator which is not consistent and an example of a consistent estimator which is biased.

22. x_1, x_2,...,x_n are independent observations of X which has density given by $f(x;\theta)=\exp\{-(x-\theta)\}$ for $x\geq\theta$, 0 otherwise. Obtain the MLE of θ. Obtain the variance of this estimator. Is it a consistent estimator of θ?

23. $X_1, X_2, \ldots, X_n$ are independent ***non-identically*** distributed random variables where X_r has density $f(x;\theta)=(r\theta)^{-1}\exp\{-x/r\theta\}$, i.e. exponential with mean $r\theta$. Investigate the unbiasedness of the following estimators of θ:

$$\frac{2(X_1+X_2+\ldots+X_n)}{n(n+1)}; \quad \frac{1}{n}\left(X_1+\frac{X_2}{2}+\ldots\frac{X_n}{n}\right); \quad \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)X_{(1)}$$

where $X_{(1)}=\min(X_1,X_2,\ldots X_n)$. Are any of these MVBU estimators?

24. The random variables X_1 and X_2 are independent and each is $N(0,\theta^2)$. Compare the relative merits of $\frac{1}{2}\sqrt{\pi}\,|X_1-X_2|$ and $\sqrt{\dfrac{2(X_1^2+X_2^2)}{\pi}}$ as estimators of θ. Are either of these MVBU estimators of θ?

25. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x;\theta)=\theta^x(1-\theta)^{1-x}$, $(x=0,1;\ 0\le\theta\le1)$. Find the method of moments estimator of θ and evaluate its mean and MSE. Obtain the m.l.e of θ and evaluate its mean and MSE.

26. In genetic investigations it is common to sample from a binomial distribution $B(m,\theta)$ except that observations of $x=0$ are impossible, thus the observations are from a truncated binomial distribution $f(x;\theta)=\dfrac{\binom{m}{x}\theta^x(1-\theta)^{m-x}}{1-(1-\theta)^m}$ $x=1,2,\ldots,m$.

    Determine the mle of θ in the case $m=2$ in samples of size n.

27. The random variables $X_1, X_2, \ldots, X_n$ are i.i.d. $N(\theta, c^2\theta^2)$, where c is known. Find the mle of θ and determine its asymptotic variance. Shew that the asymptotic relative efficiency of the sample mean is $1/(1+2c^2)$.

28. The random variables $X_1, X_2, \ldots, X_n$ are i.i.d, each with pdf

$$\frac{4x^2}{\theta^3\sqrt{\pi}}\exp\left(-\frac{x^2}{\theta^2}\right) \quad (x>0).$$

    Derive the mle of θ and find its asymptotic variace. What is the asymptotic relative efficiency of the alternative estimator defined by $\frac{\sqrt{\pi}}{2n}(X_1+X_2+\ldots X_n)$?

29. The lifetime of a certain type of particle in a cloud chamber experiment is a random variable with an exponential distribution with mean λ, but only lifetimes not exceeding a specific value K can be measured. Out of n particles, n–m had lifetimes exceeding K, and the measured lifetimes of the other m were $x_1, x_2, \ldots, x_m$. Shew that the mle of λ is

$$\frac{X_1 + X_2 + \ldots + X_m + (n-m)K}{m}$$

and evaluate its variance.

30. Supose $X_1, X_2, \ldots, X_n$ are independent observations of $X \sim P(\theta)$. Define $T(\mathbf{x})=r!$ if $X_1=r$ and 0 otherwise. Shew that $T(\mathbf{x})$ is unbiased for $\tau(\theta)=\theta^r e^{-\theta}$. Use the Rao-Blackwell theorem to improve on this estimator.

32. Suppose X_1, $X_2,\ldots,X_n$ are independent observations of $X \sim N(0,\theta^2)$. Shew that $\sqrt{\pi/2}\,|X_1|$ is unbiased for θ and use the Blackwell-Rao theorem to improve on this estimate.

31. Suppose $X_1, X_2,\ldots,X_n$ are independent observations of $X\sim P(\theta)$.

(a) Find a complete sufficient statistic for θ which has a Poisson distribution with mean $n\theta$ and hence find the minimum variance unbiased estimator of θ^r.

(b) Find the minimum variance unbiased estimator of $e^{-\alpha\theta}$ where α is a known constant. [Suggestion: find a sufficient statistic T for θ which has a Poisson distribution with mean $n\theta$. Then, let $\hat{\eta}=\hat{\eta}(T)$ be the unbiased estimator of $e^{-\alpha\theta}$ and, since it is a function of T, find an expression for its expectation which, when multiplied by $e^{-n\theta}$, is a power series in θ^t (t=0,1,....). Since this must equal $e^{(n-\alpha)\theta}$ deduce the UMV estimator by equating coefficients of θ^t.]

(c) What is the maximum likelihood estimator of $e^{-\alpha\theta}$? Why, without doing any calculations, do you know that it must be biased?

32. Suppose $x_1, x_2,\ldots,x_n$ are independent observations of X with density $f(x;\theta)=3\theta^3/x^4$; $x\geq\theta$, 0 otherwise.

(a) Shew that $X_{(1)}=\min_{1\leq i\leq n}\{X_i\}$ is sufficient for θ.

(b) Find the maximum likelihood estimator of θ. Is it consistent?

(c) Find $E[(cX_{(1)}-\theta)^2]$ for each value of c and shew that this has a minimum value at $c=(3n-2)/(3n-1)$. Deduce that under the mean squared error criterion the MLE of θ is inadmissible.

(d) Is there an unbiased estimator of θ of the form $cX_{(1)}$ which is admissible under the mean squared error criterion?

33. Suppose $x_1, x_2,\ldots,x_n$ are independent observations of $X\sim N(\mu,1)$. Shew that the best unbiased estimator of μ^2 is inadmissible.

34. Suppose $x_1, x_2,...,x_n$ are independent observations of $X \sim P(\theta)$. Using a result from Q31, shew that the best unbiased estimator of $e^{-(n+1)\theta}$ is inadmissible.

[Suggestion for Q33&34: If $\eta(\theta) \in (a,b)$ but $P[\hat{\eta} < a \text{ or } > b] > 0$ then $\hat{\eta}$ is inadmissible since it can be improved (wrt the mse criterion) by restricting (i.e. trimming) it to the range (a,b).]

35. A senior consultant at the Royal Hallamshire Hospital has come to you for advice in estimating the area (in $[\text{image units}]^2$) of circular lesions in a micrograph of a section of kidney. She assures you that (in image units) the radii of these lesions are normally distributed with unit variance about the true value. Further, the mean radius of three independent measurements from separate micrographs is 0.5 image units. Knowing that you are likely to have to provide many estimates in similar situations for her in the future, what answer to you give her?

36. Customers arrive at a shop in a Poisson process with rate θ, per ten minutes, which might vary from day to day. The shop opens at 9.00a.m. and one day the shop assistant (who opened the shop at exactly opening time) noticed that only 1 customer had arrived before the manager who came at 9.10a.m. The manager asked the assistant how business was going and in particular what were the chances of at least one customer arriving by 9.30.a.m? How should the shop assistant (who has successfully completed Q31 above) reply without running the risk of being ridiculed?

37. A machine has two possible settings 1 and 2. At an operation with setting 1 it is known to produce a component whose dimension is described by a N(10,1) r.v.; while at setting 2 the description is by a N(10,25,1) r.v. Unfortunately the setting for a long run of independent operations has not been noted.

(i) It is proposed to measure 25 of the components, if there mean dimension is under 10.1 then decide that setting 1 was used and otherwise decide it was setting 2. Obtain measures of liabilities of reaching the wrong conclusion.

(ii) How many components should be measured to ensure that this rule keeps the probability of wrongly deciding for setting 2 down to 0.01? What then is the probability of the other wrong decision?

(iii) For 100 components where should the deciding line be drawn so that the probability of a wrongful decision for setting 1 is 0.02?

38. The r.v.s X_1, X_2,...,X_n are i.i.d. and known to be either N(0, ½) or N(0,1) . To test the null hypothesis that the distribution is N(0, ½) against the alternative that it is N(0,1) three tests of size 0.05 are proposed with critical regions as follows:

(i) $\left\{\mathbf{x} : \sum_{i=1}^{n}(x_i - \bar{x})^2 > c_1\right\}$

(ii) $\left\{\mathbf{x} : \left|\sum_{i=1}^{n} x_i\right| > c_2\right\}$

(iii) $\{\mathbf{x} : \min(x_1, x_2, \ldots, x_n) < c_3\}$

Place these tests in order of preference for the case n=10. Can you suggest a test which is more powerful?

39. An experiment is described by an exponential random variable with mean θ and x_1 and x_2. A proposed test of the hypothesis $\theta=2$ against the alternative $\theta=\frac{1}{2}$ uses the critical region

$$\{(x_1,x_2): \min(x_1,x_2)<\tfrac{1}{4}\}$$

Shew that this test has size of approximately 0.22 and find its power. Obtain a better test of the same size.

40. The r^{th} of three independent experiments is described by a Nμ_r,1) r.v. and $\mathbf{x}=(x_1,x_2,x_3)$ is a set of outcomes of these experiments. Show that a most powerful test of the hypothesis H_0:$\mu_r=0$ (r=1,2,3) against the alternative H_1:$\mu_r=r$ (r=1,2,3) uses a critical region f the form $\{\mathbf{x}=(x_1,x_2,x_3): x_1+2x_2+3x_3>c\}$. Determine the power of the test when c is chosen to give a size of 0.05.

41. The r.v.s X_1 and X_2 are i.i.d. each with pdf $f(x;\theta)=(\theta/\pi x)^{-\frac{1}{2}}\exp(-\theta x)$. Shew that the Neyman-Pearson construction of a most powerful test of size α of the null hypothesis that $\theta=\theta_0$ against the alternative hypothesis that $\theta=\theta_1$ where $\theta_0>\theta_1$ provides a critical region $\{(x_1,x_2): x_1+x_2>-\theta^{-1}\ln(\alpha)\}$.

Shew that the power of this test is $\alpha^{\theta_1/\theta_0}$

42. The r.v.s $X_1,X_2,\ldots,X_n$ are independent with $X_i\sim N(\theta_I,1)$ (I=1,2,…,n). Shew that the N-P Construction of a most powerful test of size 0.05 that $\theta_I=0$ against the alternative that $\theta_I=\frac{1}{2}$ for I=1,2,…,r and $\theta_j=-\frac{1}{2}$ for j=r+1,r+2,…,n provides a critical region

$$\left\{(x_1,x_2,\ldots,x_n): \sum_{i=1}^{r} x_i - \sum_{i=r+1}^{n} x_i > 1.645\sqrt{n}\right\}$$

How large must n be to ensure that the power of the test is at least 0.9?

43. Suppose x_1 and x_2 are two independent observations of $X \sim P(\lambda)$. What is the best test of size 0.1 of H_0: $\lambda=1$ *vs.* H_1: $\lambda=2$? What would be the recommendations of the test if

    (a) $x_1=3$ and $x_2=0$ (b) $x_1=3$ and $x_2=1$?

44. Suppose $x_1, x_2, \ldots, x_n$ are independent observations of X with a Beta distribution $B(\alpha,\beta)$ with α unknown and $\beta=3$. What is the most powerful test of size 0.05 of H_0: $\alpha=1$ *vs.* H_1: $\alpha=3$?

45. Suppose $x_1, x_2, \ldots, x_n$ are independent observations of $X \sim N(\mu,\sigma^2)$ with μ unknown. Find the most powerful test of H_0: $\sigma^2=\sigma^2_0$ *vs.* H_1: $\sigma^2=\sigma^2_1$ where $\sigma^2_1>\sigma^2_0$. Is this test uniformly most powerful against alternatives $\sigma^2_1 \geq \sigma^2_0$?

46. (from Jan 1997) (a) Suppose we wish to test on the basis of n independent observations whether the mean μ of a binomial distribution on {0,1,2,3,4} is equal to 2 or less than 2. Obtain the uniformly most powerful randomized test of size 0.05 for the problem.

    (b) For each of n=2 and n=3, derive explicitly the test and the corresponding power function. (You need not simplify the expressions for the power functions that you obtain.)

    (c) For sample values of 1,0,3 what would be the recommendation of the test?

    (d) Investigate whether or not for n=5, the uniformly most powerful test obtained reduces to the following nonrandomized test approximately: "Reject H_0: $\mu=2$ if $\sqrt{5}(\overline{X}-2) \leq -1.645$".

47. The r.v.s $X_1,X_2,...,X_n$ are i.i.d. $\Gamma(k,\theta)$, where k is known…, i.e. $f(x,\theta)=\theta^k x^{k-1}e^{-k\theta}/\Gamma(k)$ for x>0. Shew that a UMP test of the null hypothesis that $\theta=\theta_0$ against the alternative $\theta>\theta_0$ ecists, and for the case k=1/n and significance level α has power function $1-(1-\alpha)^{\theta/\theta_0}$.

48. The random variables X_1 and X_2 are independent and $N(0,\theta)$. Find the critical region of size α of the most power test of H_0:$\theta=\theta_0$ against the alternative H_1:$\theta=\theta_1$ where $\theta_1>\theta_0$. What is the widest alternative hypothesis for which this critical region provides a most powerful test? Derive the likelihood ratio test at significance level α of the null hypothesis that $\theta\leq\theta_0$ against the alternative that $\theta>\theta_0$ and obtain the power function.

49. Suppose $X_1,X_2,...,X_n$ are i.i.d. $U(0,\theta)$ r.v.s.

(i) Derive the constant c of the test of size α of the null hypothesis H_0:$\theta\leq 3$ against the alternative H_1:$\theta>3$ with critical region of the form $\{\mathbf{x}: \max(x_1,x_2,...,x_n)>c\}$.

(ii) Suppose now it is desired to test H_0:$\theta=3$ against the alternative H_1:$\theta\neq 3$. Consider a test procedure with critical region

$$C=\{\mathbf{x}: \max(x_1,x_2,...,x_n)<c_1 \text{ or } \max(x_1,x_2,...,x_n)>c_2\}$$

Determine values of c_1 and c_2 such that the power function of the test $\eta(\theta)$ is such that $\eta(3)=0.05$ and $\eta(\theta)>0.05$ for $\theta\neq 3$.

50. $x_1,x_2,...,x_n$ are independent observations of X which has pdf $f(x;\theta)=\theta\lambda^\theta/x^{\theta+1}$ where λ is known and θ (>0) is an unknown parameter.

    (i) Shew that there is a UMP test of the null hypothesis that $\theta=1$ against $\theta>1$.

    (ii) Find the mle of θ and evaluate its asymptotic variance.

    Shew that there is no UMP test of $\theta=1$ against the alternative that $\theta\neq1$ but that the generalized LRT uses a critical region of the form $\left\{(x_1,x_2,...,x_n):\dfrac{\tilde{x}}{\ln(\tilde{x}/\lambda)}>c\right\}$ where $\tilde{x}=\sqrt[n]{x_1x_2...x_n}$. Assuming that the generalized LRT statistic is asymptotically χ^2 on 1df under the null hypothesis, determine the value of c above for large n.

51. $x_1,x_2,...,x_n$ are independent observations of X which has pdf $f(x;\theta)=\exp\{-(x-\theta)\}$ $(x>\theta)$. Shew that the generalized LRT of the null hypothesis $\theta\leq1$ against the alternative that $\theta>1$ use a critical region of the form $\{\mathbf{x}: \min(x_1,x_2,...,x_n)>c\}$. Determine c so that the size of this critical region is α. Sketch the graph of the power function $\eta(\theta)$ of this test.

52. $x_1,x_2,...,x_n$ are independent observations of $X\sim E(\lambda)$ (i.e. exponential with mean λ) and $y_1,y_2,\ldots,y_n$ are independent observations of $Y\sim E(\mu)$. Find the form of the critical region of the LRT of the hypothesis $\lambda=\mu$ against the alternative $\lambda\neq\mu$.

53. Suppose $x_1,x_2,...,x_n$ are independent observations of $X\sim N(\mu,\sigma^2)$ with known mean μ and unknown variance σ^2. Three possible confidence intervals for σ^2 are:–

(a) $\left(\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{a_1}, \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{a_2}\right)$, (b) $\left(\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{b_1}, \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{b_2}\right)$

(c) $\left(\frac{n(\bar{x}-\mu)^2}{c_1}, \frac{n(\bar{x}-\mu)^2}{c_2}\right)$, where a_1,a_2,b_1,b_2,c_1 and c_2 are constants.

(i) Find values of these six constants which give a confidence coefficient for each of the three intervals of 0.90 when n=10.

(ii) Compare the expected length of each of the three intervals. in this case.

(iii) Which of these intervals is the most stringent of the three?

(iv) With $\sigma^2=1$, what value of n is required to achieve a 90% confidence interval of length less than 2 in cases (b) and (c) above?

54. Suppose $x_1, x_2,...,x_n$ are independent observations of X with density $f(x;\theta)=\theta x^{\theta-1}$ for $0<x<1$ and $\theta>1$. Find the maximum likelihood estimator of θ and hence obtain an approximate $100(1-\alpha)\%$ interval for θ.

55. Suppose x_1, x_2,...,x_n are independent observations of X distributed uniformly on $(0,\theta)$. Let $x_{(n)}=\max_{1\leq i\leq n}\{x_i\}$. Find the constant k_α such that $T_\alpha=x_{(n)}k_\alpha$ is an upper $100(1-\alpha)\%$ confidence limit for θ.

    Is T_α the most stringent upper $100(1-\alpha)\%$ confidence limit for θ?

56. Suppose X is uniformly distributed on $(\theta-½,\theta+½)$ and that x_1 and x_2 are two independent observations of X. Let $x_{(1)}=\min_{1,2}\{x_i\}$ and $x_{(2)}=\max_{1,2}\{x_i\}$. By noting that $P[X_{(1)}\leq\theta\leq X_{(2)}]=P[X_1\leq\theta\leq X_2]+P[X_2\leq\theta\leq X_1]$ shew that the interval $(x_{(1)},x_{(2)})$ is a confidence interval for θ with confidence coefficient ½. If $x_1=7.0$ and $x_2=6.25$ calculate the width of this 50% confidence interval and comment on the result.