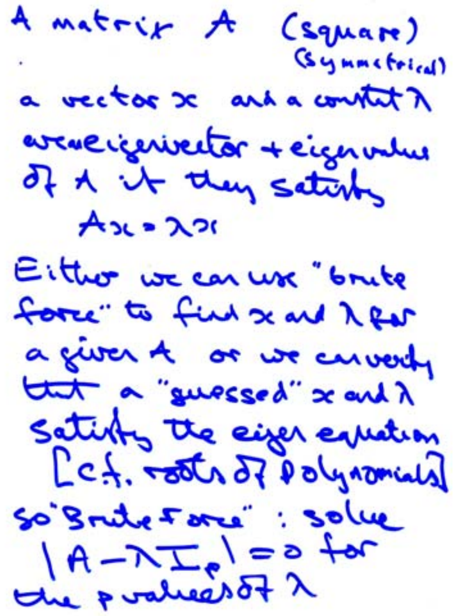
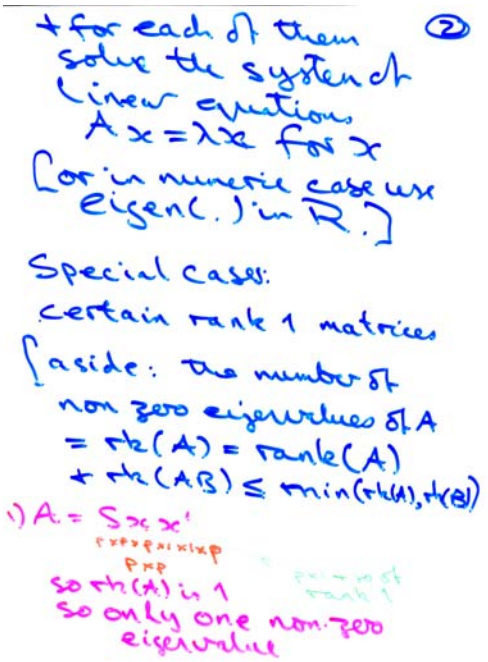
Notes from lecture 21/10/10

These are copies of the OHP transparencies from the lecture on Thursday 22/10/09 and repeated on 21/10/10. There are a couple of deceptively simple results relating to particular matrices.





If in doubt as to how to get eigenvalues and determinants etc from R then email me and I'll shew you.

Slide 1 points out that finding an explicit vector and scalar which satisfy the eigenequation means that they must be an eigenvector and eigenvalue. Just as finding a number which satisfies a polynomial equation means it must be a root of the polynomial The 'brute force' general method to find first the eigenvalues and then the eigenvectors. However, there are a few special cases where you 'can spot' the solution, though really you can only do this with experience of having seen it before (hence the idea of going through this now).

The key to the first trick is not so much that the matrix is of rank 1 but that within the product you can find a

(3) which is $x'Sx$ with eigenvector Sx because these satisfy the eigenequation:

$$Sx[x'Sx] = x'Sx \, . \, Sx$$

($1{\times}p\,p{\times}p\,p{\times}1$: $1{\times}1$ scalar)

2) very special case: $S = I_p$

xx' has eigenvector x, eigenvalue $x'x$

$$(xx')x = x'x(x)$$

(4) one eigenvector is z because

$$(\alpha I_p + \beta zz')z$$
$$= \alpha z + \beta zz'z \quad (1{\times}p\,p{\times}1)$$
$$= (\alpha + \beta z'z)z$$

so eigenvalue corresponding to z is $(\alpha + \beta z'z)$.

To find all the eigenvalues use result $|I_p + AB| = |I_n + BA|$ ($p{\times}n$, $n{\times}p$) + solve $|\alpha I_p + \beta zz' - \lambda I_p|$

$$= |(\alpha - \lambda) I_p + \beta zz'|$$

(5)

$$= (\alpha - \lambda)^p \left(I_1 + \frac{\beta}{\alpha - \lambda} z'z\right)$$

($I_1 = 1$, using result)

$$= (\alpha - \lambda)^{p-1}(\alpha + \beta z'z - \lambda)$$

so eigenvalues are $\alpha + \beta z'z$ and α with multiplicity $p-1$.

'factor' in the product that is 1×1 (i.e. a scalar) and this means we can take it out of the product and put it anywhere we like.

The very special case follows just by taking $S = I_p$ or prove directly

Next, in a similar vein, this works in just the same way, even though it isn't a rank 1 matrix — it's the same trick of spotting a 1×1 factor in the product and rearranging.

To find all of the eigenvalues the result on determinants quoted in task sheet 3 is need. A proof of this is give in the Basics of Matrix Algebra with R notes.

Note that for matrices such as $\boldsymbol{\alpha} I_p + \boldsymbol{\beta} zz'$ there is a great deal of symmetry and so it is likely that most of the eigenvalues will be equal. It is always a good bet with matrices like this that one of the eigenvectors will be proportional to 1_p and this can be looked for directly.