Appendix:
Details of marrying factors for spheroids

Four distinct possibilities for ordering and equivalence of the axes exist (throughout we require that the a axis is parallel to the direction of fall):

I:	a, a, b	$a < b$	prolate spheroid falling broadside on
II:	a, b, b	$a < b$	oblate spheroid falling broadside on
III:	a, a, b	$a > b$	oblate spheroid falling point downwards
IV:	a, b, b	$a > b$	prolate spheroid falling point downwards.

It seems likely that a preferred orientation will be rapidly established with 'broadside on' equilibrium proving more stable than 'point downwards' motion (IACWR, 1957). Thus cases I and II are of particular practical interest.

Sieve sizes

For the prolate particles, cases I and IV, with one long axis and two short ones, the ideal sieve size (i.e. the minimum size of square mesh that the particles can pass through) is clearly recorded when the particles are orientated with with their long axis perpendicular to the sieve. In this case the cross section passing through the mesh is circular and the sieve size is directly given (in an obvious notation) by $D_{S,I} = 2a$ or $D_{S,IV} = 2b$. For cases II and III the orientation attained during sieving should result in the passage of an elliptical cross section through the sieve mesh and thus both long and short diameters will play a part in determining the sieve size and the particle can just pass through the mesh when the two principal diameters are aligned with the diagonals of the square mesh. A simple geometric argument then shows that $D_{S,II} = D_{S,III} = \sqrt{2(a^2 + b^2)}$.

Second Order Approximations of Stokes' Law

Stokes' Law gives the drag D experienced by a sphere of radius a, density ρ, moving at a speed u in a medium with viscosity ν, as (Batchelor, 1970)

$$D = 6\pi a\nu u + 9\pi\rho a^2u^2/4 \tag{1}$$

(to a second order approximation). This shows that a spherical particle moving under gravity reaches a terminal, or settling, velocity u_S of

$$u_S = 2[-6\nu + \{36\nu^2 + 12\rho a^3(\rho - \rho_0)g\}^{1/2}]/9\rho a \tag{2}$$

where g is the acceleration due to gravity and ρ_0 is the density of the fluid. Thus determination of a sphere's settling velocity allows calculation of its radius a by inversion of equation **??**. This gives

$$a = \frac{27\rho u_S^2}{32(\rho - \rho_0)g} + \left[\left\{\frac{27\rho u_S^2}{32(\rho - \rho_0)}\right\}^2 + \frac{9\nu u_S}{2(\rho - \rho_0)g}\right]^{1/2} \tag{3}$$

Stokes' Diameters of Spheroids

For an ellipsoid with semiaxes a, b and c falling in a viscous fluid with axis a parallel to the direction of motion the diameter of the equivalent sphere, D_H, is given by Lamb(1959) as

$$D_H = \frac{16abc}{3\left(\chi_0 + \alpha_0 a^2\right)} \tag{4}$$

where

$$\chi_0 = abc \int_0^\infty \frac{d\lambda}{\Delta}$$

and

$$\alpha_0 = abc \int_0^\infty \frac{d\lambda}{\left(a^2 + \lambda\right)\Delta}$$

with

$$\Delta = \left\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\right\}^{1/2}$$

In the case of spheroids when $c = a$ or $c = b$, substitutions of the form $\lambda = a^2 \tan^2(\theta)$ or $\lambda = b^2 \tan^2(\theta)$ followed by $x = \cos(\theta)$ reduce the integral to a standard form (Abramowitz & Stegun, 1965). The four cases of prolate and oblate spheroids falling 'broadside on' and 'point downwards' need to be treated separately. The resulting equivalent diameters are given in Table **??** together with their sieve sizes. Thus we can express both the hydrometer and

case	semiaxes	orientation	Stokes' diameter	sieve size
I	$a, a, b\ (a < b)$	broadside	$\frac{16(b^2-a^2)}{3\left[\frac{2b^2-3a^2}{\sqrt{b^2-a^2}}\ln\{(b+\sqrt{b^2-a^2})/a\}+b\right]}$	$2a$
II	$a, b, b\ (a < b)$	broadside	$\frac{8(b^2-a^2)}{3\left[a+\frac{b^-2a^2}{\sqrt{b^2-a^2}}\cos^{-1}(a/b)\right]}$	$\sqrt{2(a^2+b^2)}$
III	$a, a, b\ (a > b)$	point down	$\frac{16(a^2-b^2)}{3\left[\frac{3a^2-2b^2}{\sqrt{a^2-b^2}}\cos^{-1}(b/a)-b\right]}$	$\sqrt{2(a^2+b^2)}$
IV	$a, b, b\ (a > b)$	point down	$\frac{8(a^2-b^2)}{3\left[\frac{2a^2-b^2}{\sqrt{a^2-b^2}}\ln\{(a+\sqrt{a^2-b^2})/b\}-a\right]}$	$2b$

Table 1: Stokes' diameters of spheroids

sieve sizes as functions of the geometrical dimensions of the particle. This allows us to ascertain their relationship with each other.

Examination of Table **??** shows that in each case I,...,IV

$$D_S \;=\; \xi D_H, \tag{5}$$

where ξ is the required marrying factor, a function of the axial ratios of the spheroid, i.e. of its *shape*. The precise form of the function depends on the orientation of the particle with respect to its motion. If this is known (by applying hydrodynamical arguments, for example), then knowledge of the marrying factor allows determination of the particle's shape.

This marrying factor, ξ, required to convert sizes determined by the hydrometer to notional sieve sizes can be expressed in terms of the axial ratios of the ellipsoid. This is an easily appreciated 'shape parameter' of the ellipsoid and it is in this form that the marrying factors are given in Table **??** below. We have taken this ratio, which we denote by γ, as a/b if $a < b$ and b/a if $a > b$ (i.e. smaller divided by larger) so that the ratio lies between 0 and 1.

case	semiaxes	orientation	γ	marrying factor
I	a, a, b $(a < b)$	broadside	a/b	$\frac{3[(2-3\gamma^2)\ln\{1/\gamma+\sqrt{1/\gamma^2-1}\}+\sqrt{1-\gamma^2}]}{8(1-\gamma^2)^{3/2}}$
II	a, b, b $(a < b)$	broadside	a/b	$\frac{3\sqrt{2(1+\gamma^2)}[\gamma\sqrt{1-\gamma^2}+(1-2\gamma^2)\cos^{-1}(\gamma)]}{8(1-\gamma^2)^{3/2}}$
III	a, a, b $(a > b)$	point down	b/a	$\frac{3\sqrt{2(1+\gamma^2)}[(3-2\gamma^2)\cos^{-1}(\gamma)-\gamma\sqrt{1-\gamma^2}]}{16(1-\gamma^2)^{3/2}}$
IV	a, b, b $(a > b)$	point down	b/a	$\frac{3[(2-\gamma^2)\ln\{1/\gamma+\sqrt{1/\gamma^2-1}\}-\sqrt{1-\gamma^2}]}{4(1-\gamma^2)^{3/2}}$

Table 2: Marrying factors for spheroids

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